Mech. Stosowanej, Vol.16, №6, 1964.
3. Il'iushin, A.A., Deformation of a visco-plastic solid. Uch. Zap. Mosk. Univ., Vyp. 39, 1940.
4. Ishlinskii, A.Iu., On the stability of visco-plastic flow in a circular plate. PMM Vol.7, №6, 1943.
5. Ishlinskii, A.Iu., On the stability of visco-plastic flow in a strip and a circular rod. PMM Vol.7, №2, 1943.
6. Sporykhin, A.N., On the strain stability of elastic visco-plastic solids. Prikl. Mekh. Tekhn. Fiz., №4, 1967.
7. Sporykhin, A.N., On the equilibrium stability of an elastic visco-plastic medium. Prikl. Mekh. Tekhn. Fiz., №5, 1970.
8. Sedov, L.I., Introduction to the Mechanics of a Continuous Medium. Fizmatgiz, Moscow, 1962.
9. Truesdell, C. and Toupin, R., The Classical Field Theories. In: Handbuch der Physik, Vol.3/1, pp.226-793, Springer Verlag, Berlin-Göttingen-Heidelberg, 1960.
10. Green, A.E., Rivlin, R.S. and Schield, R.T., General theory of a small elastic deformation superposed on finite elastic deformations. Proc. Roy. Soc. A, Vol.211, №1104, 1957.
11. Rabotnov, Iu.N. and Shesterikov, S.A., Stability of bars and plates under creep conditions. PMM Vol.21, №3, 1957.
12. Ivlev, D.D. and Bykovtsev, G.I., Theory of a Hardening Plastic Solid. "Nauka", Moscow, 1971.
13. Guz'A.E., Stability of Three-dimensional Deformable Bodies. "Naukova Dumka", Kiev, 1971.
14. Bolotin, V.V., Nonconservative Problems of the Theory of Elastic Stability. (English translation), Pergamon Press, Book №10490, 1963.
15. Biot, M.A., Surface instability in finite anisotropic elasticity under initial stress. Proc. Roy. Soc. A Vol.273, №1354, 1963.
16. Biot, M.A., Internal buckling under initial stress in finite elasticity. Proc. Roy. Soc. A Vol.273, №1354, 1963.

Translated by M.D.F.

UDC 539.375

INFLUENCE OF A RIGID INCLUSION ON THE STRESS INTENSITY NEAR THE TIPS OF A CRACK

PMM Vol.38, №4, 1974, pp.719-727

G.T.ZHORZHOLIANI and A.I.KALANDIIA
(Tbilisi)

(Received August 13, 1973)

The stress intensity factor in a plate containing a rigid circular inclusion is determined by reduction to an integral equation with a Cauchy kernel and finding its numerical solution.

1. An elastic medium occupies the whole ($z = x + iy$)-plane with a circular hole

of unit radius and a radial crack of length $2l$ on the x-axis as is shown in Fig. 1. The hole is filled by an absolutely rigid core welded to the matrix along the outline. The edges of the crack are free of external forces, the medium is subject to tensile forces P at infinity, perpendicular to the line of the crack and there is no rotation at infinitely remote parts of the plane. Let γ be the circumference of the hole, and $L = (a, b)$ the crack without its tips. Let us distinguish the upper and lower edge of the crack and ascribe quantities to them to which the plus and minus signs, respectively, refer.

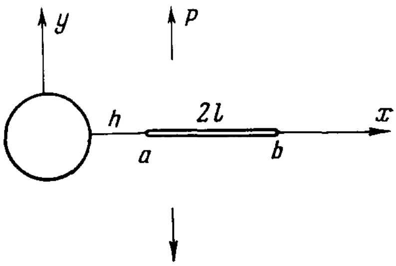

Fig. 1

The problem is to determine the stresses and displacements in an elastic body, subject to the boundary conditions which express the absence of elastic displacements along the hole circumference and of external forces along the crack

$$u = v = 0 \quad \text{on } \gamma, \qquad \sigma_y^{\pm} = \tau_{xy}^{\pm} = 0 \quad \text{on } L \tag{1.1}$$

Let us introduce the complex potentials $\varphi(z)$, $\psi(z)$ and let us use the notation

$$\omega(z) = \frac{\varphi(z)}{z} + \psi(z)$$
$$\Omega(z, \bar{z}) = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = \varphi(z) + \overline{\omega(z)} + \left(z - \frac{1}{\bar{z}}\right)\overline{\varphi'(z)} \tag{1.2}$$

Then according to the known Kolosov-Muskhelishvili representations [1]

$$2\mu(u + iv) = (\varkappa + 1)\varphi(z) - \Omega(z, \bar{z}) \tag{1.3}$$

$$\sigma_x + \sigma_y = 2[\varphi'(z) + \overline{\varphi'(z)}], \quad \sigma_y - \sigma_x - 2i\tau_{xy} = 2\frac{\partial}{\partial \bar{z}}\Omega(z, \bar{z}) \tag{1.4}$$

where $\varkappa = 3 - 4\nu$, ν is the Poisson's ratio. We write the boundary conditions of the problem as

$$\varkappa\varphi(\sigma) - \overline{\omega(\sigma)} = 0 \quad \text{on } \gamma\ (\sigma = e^{i\theta}) \tag{1.5}$$

$$\sigma_y^{\pm} - i\tau_{xy}^{\pm} = \frac{d}{dt}\Omega^{\pm}(t, t) = 0 \quad \text{on } L\ (a < t < b) \tag{1.6}$$

2. Let us represent the solution as the sum of two components

$$\varphi(z) = \varphi_*(z) + \varphi_0(z)$$
$$\omega(z) = \omega_*(z) + \omega_0(z) \quad (\psi(z) = \psi_*(z) + \psi_0(z)) \tag{2.1}$$

where φ_0, ψ_0 yield the solution of the problem of a rigid inclusion without a crack and φ_*, ψ_* characterize the additional field.

The functions φ_0, ω_0 are determined from condition (1.5) and are found easily by the Muskhelishvili method. The solution of the second fundamental plane problem (elastic displacements given on the boundary) for the exterior of a circle is given by formulas (3) and (4) in Sect. 83 of the monograph [1] if we set $m = 0$.

$$\varkappa\varphi(\sigma) - \overline{\omega(\sigma)} = f(\sigma) \quad \text{on } \gamma \tag{2.2}$$

under the condition that there are no stresses and rotation at infinity as well as that the principal vector of the external forces applied to the boundary of the medium equals zero.

According to these formulas, if it is taken into account that for large $|z|$

$$\varphi_0(z) = Az + \varphi_0^*(z), \qquad \omega_0(z) = Bz + \omega_0^*(z) \qquad A = P/4, \qquad B = P/2 \tag{2.3}$$

where φ_0^*, ω_0^* are regular functions everywhere outside the circular hole, we find by discarding the inessential constant in the formulas mentioned

$$\varphi_0(z) = Az + B/\varkappa z, \qquad \omega_0(z) = Bz + \varkappa A/z, \qquad |z| > 1 \tag{2.4}$$

Finding φ_*, ψ_* is much more complicated, it is equivalent to solving the formulated problem. From the boundary condition (1.5) we determine the analytic functions $\varphi_*(z, t)$, $\omega_*(z, t)$, dependent on the point t from the interval (a, b), which are regular everywhere outside the hole except at points of the interval itself, and admit the following representation in the neighborhood of $z = t$:

$$\varphi_*(z, t) = -p(t)\ln(z-t) + \varphi_*^\circ(z, t) \tag{2.5}$$
$$\omega_*(z, t) = -p(t)\left[\ln(z-t) + \frac{1-zt}{z(z-t)}\right] + \omega_*^\circ(z, t)$$

Here $p(t)$ is a still arbitrary real function of t defined on L. The functions φ_*°, ω_*° should be regular everywhere for $|z| > 1$, including the infinitely remote point, for any t from L. To determine them, we have in conformity with (1.5)

$$\varkappa\varphi_*^\circ(\sigma, t) - \overline{\omega_*^\circ(\sigma, t)} = f_0(\sigma, t) \tag{2.6}$$
$$f_0(\sigma, t) = p(t)\left[\varkappa\ln(\sigma-t) - \ln\left(\frac{1}{\sigma} - t\right) - \frac{\sigma(\sigma-t)}{1-\sigma t}\right]$$

We have again arrived at the problem (2.2) with a right side f defined by the previous equality. Solving it by means of the same formulas [1], we find

$$\varkappa\varphi_*^\circ(z, t) = -p(t)\left[\ln\left(1-\frac{1}{zt}\right) - \frac{1-t^2}{t^2(zt-1)}\right] \tag{2.7}$$
$$\omega_*^\circ(z, t) = -p(t)\left[\varkappa\ln\left(1-\frac{1}{zt}\right) + \frac{1}{zt}\right], \quad |z| > 1$$

Now, let us set

$$\varphi_*(z) = \frac{1}{\pi}\int_L \varphi_*(z, t)\,dt = -\frac{1}{\pi}\int_L K_1(z, t)\,p(t)\,dt \tag{2.8}$$
$$\omega_*(z) = \frac{1}{\pi}\int_L \omega_*(z, t)\,dt = -\frac{1}{\pi}\int_L K_2(z, t)\,p(t)\,dt$$

Here $K_1(z, t)$ and $K_2(z, t)$ are determined on the basis of (2.5) and (2.7). The potentials $\varphi(z)$ and $\omega(z)$ in the total field will be obtained in conformity with (2.1) by adding the two composite potentials (2.4) and (2.8).

To clarify the mechanical meaning of the function $p(x)$ $(a < x < b)$, let us examine the expression

$$K(x, t) = K_1(z, t) + \overline{K_2(z, t)} + \left(z - \frac{1}{\bar{z}}\right)\overline{\frac{\partial K_1(z, t)}{\partial z}} \qquad \text{for } z = x$$

On the basis of (2.5) and (2.7), we find

$$K(x,\ t) = \ln(x - t) + \overline{\ln(x - t)} + K_0(x,\ t) \tag{2.9}$$

$$K_0(x, t) = \frac{\varkappa^2 + 1}{\varkappa}\ln\left(1 - \frac{1}{xt}\right) + \left(\frac{1 - t^2}{t^2} + \frac{1 - x^2}{x^2}\right)\frac{1}{\varkappa(1 - xt)} - \frac{(1 - t^2)(1 - x^2)}{\varkappa xt(1 - xt)^2} + \frac{xt + 1}{xt}$$

It follows from the preceding formulas and the representation (2.8) that the combination

$$\Omega_*(z,\ \bar{z}) = \varphi_*(z) + \overline{z\varphi_*'(z)} + \overline{\psi_*(z)}$$

is real on the x-axis, i.e.

$$\operatorname{Im}\Omega_*(x,\ x) = 0, \quad |x| \geqslant 1 \tag{2.10}$$

Differentiating (1.3) with respect to x, we find

$$2\mu\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = (\varkappa + 1)\varphi'(z) - \frac{\partial}{\partial x}\Omega(z, \bar{z}) \tag{2.11}$$

The potentials (2.4) are real on the x-axis, hence, the vertical displacement v and the shear stress τ_{xy} corresponding to these potentials are zero everywhere on the x-axis. Hence, there follows from (2.10) and (2.11) that

$$2\mu\frac{\partial v}{\partial x} = (\varkappa + 1)\operatorname{Im}\varphi'(z) \quad \text{for } z = x\,(|x| > 1, x \neq a, b)$$

From this and from (2.8) we obtain for points x lying on the crack L

$$\pm p(x) = \frac{2\mu}{\varkappa + 1}\frac{dv^{\pm}}{dx} \quad \text{on } L \tag{2.12}$$

by using the Sokhotskii-Plemelj formula.

Note. The crux of the method applied is the construction of the potentials (2.8) corresponding to the unknown normal displacements along the crack in explicit form, which will result, as will be shown below, in a singular equation of the first kind in the desired dislocation density $p(x)$.

Bueckner [2, 3] indicated the method. As an idea it is contained also in the paper [4], published almost simultaneously with [2], which refers to the bending of semicircular plates. The physical meaning of the function $p(x)$ can be diverse. In [4] it is defined in the interval $(-1, 1)$, is complex valued and is represented as a generalized load concentrated at a point of the interval, but in the case under consideration, the integral of this function on the segment $[a, x]$ is treated as a normal displacement also "concentrated" at a point of the crack L, as is evident from the Bueckner formula (2.12) (see [3], p. 208, formula (2.6)).

In [3], devoted to the determination of the stress field in a rotating circular ring with a radial crack at the inner boundary, the solution is carried out to the end in the case when the outer circumference of the ring is removed to infinity. The solution in closed form (see [1], Sect. 82) is not used therein in the analysis of the auxiliary problem analogous to (2.5), (2.6), which complicates the procedure of finding formulas of the form (2.7) somewhat.

The problem of an elastic circular inclusion in a medium with an isolated crack (*)

*) The author of [5] was apparently not acquainted with the Bueckner papers.

is considered by the same method in [5]. This note includes the case both of a rigid inclusion and a cavity, and the crack was permitted to reach the edge of the inclusion.

Finally, let us note that if there is no crack, then understandably $v^+ = v^- =$ const on the x-axis, and in conformity with (2.12) the potentials (2.8) vanish.

3. By construction the potentials (2.4), (2.8) satisfy the condition of no elastic displacements on γ for any $p(x)$. Moreover, as has already been remarked, they are real for real z and, hence do not yield non-zero shear stresses on the x-axis (in particular along the crack) on the basis of (1.4). Therefore, there remains just to satisfy the first of the conditions (1.1) on L, which becomes

$$d / dx \, [\Omega_*(x, x) + \Omega_0(x, x)] = 0 \quad \text{on } L \tag{3.1}$$

on the basis of (1.5), where Ω is defined by the second expression in (1.2) and the functions φ, ω contained therein are given by (2.8) and (2.4). In other words, in order to satisfy all the conditions of the problem, it is sufficient to equate the normal stress σ_y, calculated for points of the crack, to zero in the total field. On the basis of the preceding formulas, (3.1) becomes

$$d/dx \left\{ -\frac{1}{\pi} \int_L K(x, t) \, p(t) \, dt + g(x) \right\} = 0 \quad \text{on } L \tag{3.2}$$

$$g(x) = (2A + B)x + \frac{(\varkappa - 1) A}{x} + \frac{B}{\varkappa} \frac{1}{x^3}$$

Here $K(x, t)$ is defined by (2.9), and A and B have the values given by (2.3).

The formula $\displaystyle \frac{d}{dx} \frac{1}{\pi} \int_L \ln(x - t) \, p(t) \, dt = -ip(x) + \frac{1}{\pi} \int_L \frac{p(t) \, dt}{x - t}$

which is valid for any function $p(x)$ continuous on L in the Hölder sense, should be used to evaluate the singular part of the kernel in (3.2). The regular part of the kernel in (3.2) can be obtained by differentiating under the integral sign. Elementary calculations based on the considerations mentioned will reduce (3.2) to the following final form:

$$\frac{1}{2\pi} \int_L \frac{p(t) \, dt}{t - x} + \frac{1}{2\pi} \int_L k_0(x, t) \, p(t) \, dt = f_0(x) \quad \text{on } L \tag{3.3}$$

$$k_0(x, t) = -\frac{1}{2} \frac{\partial}{\partial x} K_0(x, t) = \frac{1}{2\varkappa} \left\{ \left(\varkappa^2 + 1 + \frac{2}{x^2} \right) \frac{1}{x(1 - xt)} + \right. \tag{3.4}$$

$$\left. \left[\frac{t^2 - 1}{t^2} \left(2 + \frac{1}{x^2} \right) + \frac{x^2 - 1}{x^2} \right] \frac{t}{(1 - xt)^2} + \frac{2(t^2 - 1)(x^2 - 1)}{x(1 - xt)^3} + \frac{\varkappa}{x^2 t} \right\}$$

$$f_0(x) = -\frac{1}{4} g'(x) = -\frac{P}{4} \left(1 - \frac{\varkappa - 1}{4} \frac{1}{x^2} - \frac{3}{2\varkappa} \frac{1}{x^4} \right)$$

The singular equation of the first kind (3.3) is indeed the fundamental relationship of the problem. By the linear transformation of variables

$$x = l(\xi + 1) + h + 1, \qquad t = l(\eta + 1) + h + 1 \tag{3.5}$$

transforming a crack L with tips $a = 1 + h$, $b = 1 + h + 2l$ into the segment $[-1, 1]$, Eq. (3.3) is converted to a more convenient form for finding its approximate solution. The converted equation has the standard form

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{\tau(\eta)\,d\eta}{\eta-\xi}+\frac{1}{2\pi}\int_{-1}^{1}k(\xi,\eta)\,\tau(\eta)\,d\eta=f(\xi),\quad -1<\xi<1 \tag{3.6}$$

Here

$$\tau(\xi)=p(x),\qquad k(\xi,\eta)=l\,k_0(x,t),\qquad f(\xi)=f_0(x) \tag{3.7}$$

The case when the cavity in an elastic body is not filled, is obtained from the substitution $\varkappa=-1$ considered above in all the formulas (with the exception of those which contain the elastic displacements). The fundamental equation of the problem will have the previous form (3.3) for the following values of the kernel and the free term:

$$k_0(x,t)=\frac{x^2+1}{x^3(xt-1)}-\left[\frac{x^2-1}{2}+\frac{(2x^2+1)(t^2-1)}{2t^2}\right]\frac{t}{x^2(xt-1)^2}+ \tag{3.8}$$

$$\frac{(t^2-1)(x^2-1)}{x(xt-1)^3}+\frac{1}{2x^2t}$$

$$f_0(x)=-\frac{P}{4}\left(1+\frac{1}{2x^2}+\frac{3}{2}\frac{1}{x^4}\right)$$

The relationship (2.12) remains unchanged.

This case has also been considered in [3]. The same problem about a crack emerging at the free edge of a circular cavity was studied earlier by using the Mellin transform [6].

4. The normal stresses σ_x and σ_y pass into an infinity of order $^1/_2$ in the neighborhood of both tips for $h>0$. More exactly, because of the symmetry of the stress field relative to the x-axis, the relationships

$$\sigma_x=\sigma_y=\frac{N}{\sqrt{|x-c|}}+O(1),\qquad \tau_{xy}=0 \tag{4.1}$$

are valid for points of this axis outside the segment $[a,b]$ and near one of the tips c, where N is the stress intensity factor at this tip (*). For points of x near one of the tips c and on the crack line

$$\frac{dv^+}{dx}=\pm\frac{\varkappa+1}{4\mu}\,\frac{N}{\sqrt{|x-c|}}+O(\sqrt{|x-c|})$$

where the upper sign is taken for $c=a$ and the lower for $c=b$. Comparing the previous equality with (2.12), we find

$$2\sqrt{|x-c|}\,p(x)=\pm N+O\,|x-c|)$$

Hence, on the basis of (3.5), (3.7)

$$N=\pm2\sqrt{l}\lim_{\xi\to\mp1}\sqrt{1\pm\xi}\,\tau(\xi) \tag{4.2}$$

For $h>0$ (h is the distance between the left end of the crack and the boundary of the medium), independently of the kind of boundary conditions given along the hole edge, the number N is evidently nonzero for both tips. Two basic cases should be distinguished for $h=0$.

1. Hole filled with a rigid core. In the case under consideration, when

*) The number $K=\sqrt{2}N$ is often called the stress intensity factor.

the crack reaches the edge of a rigid inclusion, the order of the singularity at the appropriate tip $(x = a)$ depends on the Poisson's ratio of the material ν according to the investigation of Williams [7], and equals approximately $^1/_3$ for $\nu = 0.3$. The stresses near the tip remain however infinite, but the stress intensity factor is zero in the sense of (4.1).

2. Hole not filled at all, i.e. the crack reaches the edge of a circular cavity free of external forces. In this case, the tip $x = a$ understandably drops out; the function $p(x)$ is bounded in the neighborhood of $x = a$ and has a singularity of order $^1/_2$ at the other tip (*).

5. Let us use the method for the approximate solution of (3.6) indicated in [8]. In conformity with the above, the solution of (3.6) should be sought which is unbounded at the tips of the segment. The exception is the second case in Sect. 4.

In the first case the solution $\tau(\xi)$ is represented as

$$\tau(\xi) = \frac{\tau_0(\xi)}{\sqrt{1-\xi^2}} \tag{5.1}$$

where the function $\tau_0(\xi)$ is replaced by the Lagrange interpolation polynomial L_n constructed at the Chebyshev nodes

$$L_n[\tau_0; \xi] = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} \tau_0(\xi_k) \frac{\cos n\vartheta \sin \vartheta_k}{\cos\vartheta - \cos\vartheta_k}, \quad \xi = \cos\vartheta \tag{5.2}$$

$$\xi_m = \cos\vartheta_m, \qquad \vartheta_m = \frac{2m-1}{2n}\pi, \qquad m = 1, 2, \ldots n \tag{5.3}$$

The method reduces Eq. (3.6) to a system of linear equations

$$\sum_{\nu=1}^{n} \alpha_{m\nu}\tau_\nu^\circ = f_m, \qquad m = 1, 2, \ldots, n \tag{5.4}$$

$$\alpha_{m\nu} = \frac{1}{2n}\left[\frac{1}{\sin\vartheta_m}\operatorname{ctg}\frac{\vartheta_m \pm \vartheta_\nu}{2} + k(\xi_m, \eta_\nu)\right] \tag{5.5}$$

$$f_m = f(\xi_m), \quad \tau_m^\circ = \tau_0(\xi_m), \quad \xi_m = \eta_m$$

The functions $k(\xi, \eta)$ and $f(\xi)$ are defined by (3.7) and (3.4). The upper sign in the formula for $\alpha_{m\nu}$ is taken for $|m - \nu| = 0, 2, \ldots$, and the lower for $|m - \nu| = 1, 3, \ldots$.

After having found the quantities τ_k°, the approximate values of the desired τ_0 at the nodes (5.3), from (5.4) the stress intensity factor N is determined on the basis of (4.2), (5.1), (5.2) by the formulas

$$\begin{aligned} N_a &= \sqrt{\frac{l}{2}}\,\frac{2}{n}\sum_{k=1}^{n}(-1)^{k+n}\tau_k^\circ \operatorname{tg}\frac{\vartheta_k}{2} \\ N_b &= \sqrt{\frac{l}{2}}\,\frac{2}{n}\sum_{k=1}^{n}(-1)^{k}\tau_k^\circ \operatorname{ctg}\frac{\vartheta_k}{2} \end{aligned} \tag{5.6}$$

*) As has been mentioned above, this case has been considered in detail in [3, 6]. Here, as for the case of a rigid inclusion, we indicate a sufficiently simple and effective method of calculating the stress intensity factor for it.

In the second case, when the solution of (3.6) bounded at the left endpoint is required, we proceed from the representation

$$\tau(\xi) = \sqrt{\frac{1+\xi}{1-\xi}}\, \tau_0(\xi)$$

and as before replace $\tau_0(\xi)$ by the polynomial (5.2). We again arrive at the system (5.4) with the elements

$$\alpha_{m\nu} = \frac{1}{2n}\left[1 + \operatorname{ctg}\frac{\vartheta_m}{2} \operatorname{ctg}\frac{\vartheta_m \pm \vartheta_\nu}{2} + (1+\eta_\nu)\, k(\xi_m, \eta_\nu)\right], \; f_m = f(\xi_m) \quad (5.7)$$

The functions $k(\xi, \eta)$ and $f(\xi)$ are given this time by (3.7) and (3.8). The selection rule for the signs in (5.7) remains as before. The stress intensity factor (at the right tip of the crack) is determined by the formula

$$N_c = \sqrt{\frac{l}{2}}\, \frac{4}{n} \sum_{k=1}^{n} (-1)^k \tau_k^\circ \operatorname{ctg}\frac{\vartheta_k}{2} \tag{5.8}$$

Because of the approximate equalities $\tau_0(-1) = \tau_n^\circ$, $\tau_0(1) = \tau_1^\circ$ which are exact enough for large n, (5.6), (5.8) can be replaced by the simpler ones

$$N_a = 2\tau_n^\circ \frac{N_0}{P}, \quad N_b = -2\tau_1^\circ \frac{N_0}{P}, \quad N_c = -4\tau_1^\circ \frac{N_0}{P} \quad \left(N_0 = \sqrt{\frac{l}{2}}\, P\right)$$

Here N_0 is the stress intensity factor at the crack tips in a medium without an inclusion.

Therefore, if the number of nodes n is sufficiently large, it is then sufficient to know the values of the desired function $\tau_0(\xi)$ at the outer nodes of the interval to determine the stress intensity factor.

It should still be kept in mind that the solution of the system (5.4) with the matrix (5.5) is generally unstable. In order to make it stable, it is necessary to set an additional condition resulting from the physical meaning of the problem; in this case the evident equality

$$\int_L p(x)\, dx = l \int_{-1}^{1} \tau(\xi)\, d\xi = 0$$

which we shall take in the discrete form

$$\tau_1^\circ + \tau_2^\circ + \ldots + \tau_n^\circ = 0 \tag{5.9}$$

is the additional condition. This means that (5.9) should be appended to the system (5.4) when solving (3.6) in the class (5.1).

Values of the ratio $k = N / N_0$ as a function of the geometric parameters of the problem for $\nu = 0.3$ were determined on the M-220 electronic computer.

As should have been expected, the rigid inclusion diminishes the stress at both tips of the crack, where its influence is quite substantial for small h. As the distance h diminishes from 1 to 0.01, the ratio k decreases from 0.9266 to 0.1158 at the left tip when $l = 1$, and from 0.9711 to 0.8784 at the right tip. The decrease at the right tip for a given l becomes most significant when the left tip is in the rigid core ($h = 0$). The number k grows as the dimension l increases and varies between 0.878 and 0.988 as the latter is varied from 1 to 9.

In the case of a crack with left tip emerging on the free surface of a circular cavity

$(h = 0)$, the reverse dependence is observed. As l diminishes from 5.0 to 0.05 the ratio k increases from 2.1386 to 4.2586.

It should be noted that in this (single) case the condition of uniqueness of the displacements should be discarded. As is seen from (2.8) and (2.12), the vertical displacement v takes on an increment of $\lim\limits_{x\to 1}[v^+(x) - v^-(x)]$, equal to the width of the gap being formed at $x = 1$ after deformation, when the circle $|z| = R$, $R > 1 + 2l$ is traversed in the positive direction. In order to construct a single-valued displacement field, (2.8) should be replaced by the representation in [2, 3] and the solution of the corresponding singular equation of the form (3.6), bounded at both tips of the crack, should be determined. The method of finding such a solution is indicated in [8] cited sbove.

The system (5.4) was solved for different values of the order n up to $n = 60$. It is noteworthy that the values of k do not change substantially starting with $n = 20$.

Programing the algorithm and all the calculations needed were performed by N.N. Dzhgarkava, to whom the authors are deeply grateful.

REFERENCES

1. Muskhelishvili, N.I., Some Fundamental problems of the Mathematical Theory of Elasticity. "Nauka", Moscow, 1966. (See also English translation, Groningen, Noordhof, 1953).

2. Bueckner, H.F., Some stress singularities and their computation by means of integral equations. In: Boundary Problems in Differential Equations. Univ. of Wisconsin Press, Madison, pp. 215-230, 1960.

3. Bueckner, H.F. and Giaever, I., The stress concentration of a notched rotor subjected to centrifugal forces. ZAMM, Vol. 46, № 5, 1966.

4. Kalandiia, A.I., On a method of solving elasticity theory problems for a semicircle. In: Problems of the Mechanics of a Continuous Medium. USSR Academy of Science Press, Moscow, 1961.

5. Atkinson, C., The interaction between a crack and an inclusion. Intern. J. Engng. Sci., Vol. 10, № 2, 1972.

6. Wigglesworth, L.A., Stress relief in a cracked plate. Mathematics, Vol. 5, № 9, 1958.

7. Williams, M.L., Stress singularities resulting from various boundary conditions in angular corners of plates in extension. J. Appl. Mech., Vol. 19, № 4, 1952.

8. Kalandiia, A.I., On the approximate solution of a class of singular integral equations. Dokl. Akad. Nauk SSSR, Vol. 125, № 4, 1959.

Translated by M.D.F.